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On the Nonsymmetric, Gravitational Collapse of an Infinite, Dust-Filled Cylinder

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Abstract

Nonsymmetric terms or angle-dependent terms in the power series expansion of Einstein's equations are considered. It is shown, first, that they do not influence the time evolution of the symmetric terms and, second, that they do not remain bounded as the cylinder collapses. However, the major contribution to the density arising from the symmetric terms makes the nonsymmetric contribution progressively more insignificant as the collapse proceeds.

1. *Introduction*

The problem of small nonspherical perturbations in the gravitational field of a spherically symmetric, collapsing body has been investigated in a number of papers (Ginzburg & Ozernoy, 1964; Doroskevich *et al.*, 1965; de la Cruz *et al.*, 1970; Price, 1970). All of them show that small perturbations are radiated away as the body collapses. Novikov (1969) proved that the small perturbations of the metric remain small at the surface of a collapsing sphere. Thorne (1971) gives a review of these investigations.

Nonsymmetric terms in the metric near the center of a collapsing cylinder are discussed in this paper without assuming that they are small. Einstein's equations are studied using a power series expansion of the metric (Pachner, 1970, 1971; Miketinac & Pachner, 1972). In Section 2 equations for the coefficients of the expansion are derived. These equations are discussed in general and for a specific example in Section 3. It is shown that an initially finite, angle-dependent contribution to the density eventually becomes infinite. This contribution is absorbed in a faster increasing contribution to the density arising from symmetric terms. Thus, nonsymmetries become less and less important as the cylinder collapses.

2. *Equations of Motion*

The special, comoving coordinate system $x^\mu = (z, r, \phi, t)$ introduced by Pachner (1971) is used and his notation is followed in this paper with one exception, the function n is replaced by

$$W = 3\kappa n$$

Since no symmetry is assumed components of the metric

$$ds^2 = P^2\,dz^2 + Q^2\,dr^2 + S^2\,d\phi^2 - dt^2 + 2N\,d\phi\,dr - 2A\,d\phi\,dt$$

are functions of r, ϕ, t and their expansions are

$$\begin{aligned}
P &= \exp\left[\alpha(t) + \tfrac{1}{2}r^2 h(\phi, t) + \tfrac{1}{4}r^4\,\theta(\phi, t) + \ldots\right]\\
Q &= \exp\left[\beta(t) + \tfrac{1}{2}r^2 q(\phi, t) + \tfrac{1}{4}r^4\,\Pi(\phi, t) + \ldots\right]\\
S &= r\exp\left[\beta(t) + \tfrac{1}{2}r^2 s(\phi, t) + \tfrac{1}{4}r^4\,\Sigma(\phi, t) + \ldots\right]\\
N &= \tfrac{1}{3}r^3\,W(\phi, t) + \ldots\\
A &= \kappa r^2 \exp\left[r^2 b(\phi) + \ldots\right]\\
\psi &= \exp\left[r^2 k_2(\phi) + \ldots\right]
\end{aligned} \tag{2.1}$$

The last function determines the initial distribution of the density. Substituting equations (2.1) into Einstein's field equations

$$R_{ik} = 8\pi(T_{ik} - \tfrac{1}{2}Tg_{ik})$$

and the initial conditions

$$R_\mu{}^4 - \tfrac{1}{2}R\delta_\mu{}^4 = 8\pi T_\mu{}^4$$

one obtains equations for the coefficients in the expansion. Not all of these equations need to be considered. For a discussion of the nonsymmetric terms the following equations are important

$$\ddot{\alpha} + 2\dot{\alpha}\dot{\beta} + \dot{\alpha}^2 - 2h\,e^{-2\beta} = e^{-\alpha-2\beta} \tag{F.11.1}$$

$$\ddot{\beta} + 2\dot{\beta}^2 + \dot{\alpha}\dot{\beta} - (h - q + 3s + \tfrac{1}{2}q_{33})\,e^{-2\beta} + (W_3 - \kappa^2)\,e^{-4\beta} = e^{-\alpha-2\beta} \tag{F.22.1}$$

$$h_3 = 0 \tag{F.23.1}$$

$$\begin{aligned}
\dot{h} + (\dot{\alpha} - \dot{\beta})\,h - \tfrac{1}{2}\dot{q} + \tfrac{3}{2}\dot{s} - e^{-2\beta}[\tfrac{1}{6}\dot{W}_3 - \tfrac{1}{3}\dot{\beta}W_3 + \kappa^2(\dot{\alpha} + \dot{\beta})&\\
+ \kappa(-2b_3 - h_3 + \tfrac{1}{2}q_3 + \tfrac{1}{2}s_3)] = 0&
\end{aligned} \tag{I.2.1}$$

$$-\dot{q}_3 + e^{-2\beta}[8\kappa(b - s) - \kappa q_{33} + \tfrac{4}{3}(\dot{W} - 2\dot{\beta}W)] + 2\kappa W_3\,e^{-4\beta} = 4\kappa\,e^{-\alpha-2\beta} \tag{I.3.1}$$

$$\dot{\beta}^2 + 2\dot{\alpha}\dot{\beta} + e^{-2\beta}(-2h + q - 3s - \tfrac{1}{2}q_{33}) + W_3\,e^{-4\beta} = 2e^{-\alpha-2\beta} \tag{I.4.1}$$

$$\begin{aligned}
\tfrac{1}{2}\ddot{h} + (\dot{\alpha} + \dot{\beta})\,\dot{h} + \tfrac{1}{2}\dot{\alpha}(\dot{q} + \dot{s}) + e^{-2\beta}[-4\theta - \tfrac{1}{4}\theta_{33} - h^2 + 3qh - sh&\\
- \kappa^2(\ddot{\alpha} + \dot{\alpha}^2 + \dot{\alpha}\dot{\beta}) - \tfrac{1}{2}\kappa\dot{\alpha}(s_3 - q_3 - 2b_3)]&\\
+ e^{-4\beta}(-\kappa^2 h + \tfrac{1}{3}hW_3 - \tfrac{4}{3}\kappa\dot{\alpha}W)&\\
= -\tfrac{1}{2}e^{-\alpha-2\beta}(-2k_2 + h + q + s + \kappa^2\,e^{-2\beta})&
\end{aligned} \tag{F.11.2}$$

and

$$\tfrac{1}{2}\dot{W} + (\tfrac{1}{2}\dot{\alpha} - \dot{\beta})\,\dot{W} + 2\dot{\beta}^2 W - 3\kappa(\dot{s} + \dot{\alpha}s - b\dot{\alpha}) - \frac{9}{4}\theta_3 + \tfrac{3}{2}hq_3$$
$$+ e^{-2\beta}[3\kappa^2(\tfrac{1}{2}q_3 + \tfrac{1}{2}s_3 - 2b_3) - W(h - q + 3s + \tfrac{1}{2}q_{33})$$
$$- \kappa^2(\dot{\beta}W_3 - \tfrac{1}{3}\dot{W}_3)] + W(W_3 - \kappa^2)\,e^{-4\beta} = W e^{-\alpha-2\beta} \quad (\text{F.23.2})$$

Since α and β do not depend on ϕ, it follows from (F.11.1) that h cannot depend on ϕ; this is also expressed by (F.23.1). Similarly, from (F.22.1) one concludes that q, s, and W depend on ϕ in the following way

$$q(\phi, t) = q(t) + \bar{q}(\phi, t)$$
$$s(\phi, t) = s(t) + \bar{s}(\phi, t)$$
$$W(\phi, t) = W(t) + \bar{W}(\phi, t)$$

where

$$\bar{q} - 3\bar{s} - \tfrac{1}{2}\bar{q}_{33} + \bar{W}_3\,e^{-2\beta} = 0$$

This equation could also have been obtained from (I.4.1). In the same way it follows

$$\theta(\phi, t) = \theta(t) + \bar{\theta}(\phi, t)$$
$$b(\phi) = b + \bar{b}(\phi)$$
$$k_2(\phi) = k_2 + \bar{k}_2(\phi)$$

The unbarred functions, ϕ independent, are referred to as the symmetric terms; the barred functions are the nonsymmetric terms. Equations (F.11.1) to (F.23.2) divide into two sets of equations, one for the symmetric terms and one for the nonsymmetric terms. Equations for the unbarred functions were discussed extensively by Pachner (1971) and Miketinac & Pachner (1972). These equations contain no barred functions and, therefore, the time evolution of the symmetric terms is completely independent of the nonsymmetric terms.

Assuming solutions of the form

$$\bar{q} = A_q \cos 2\phi + B_q \sin 2\phi$$
$$\bar{s} = A_s \cos 2\phi + B_s \sin 2\phi$$
$$\bar{W} = A_W \sin 2\phi + B_W \cos 2\phi$$
$$\bar{b} = A_b \cos 2\phi + B_b \sin 2\phi$$
$$\bar{k}_2 = A_k \cos 2\phi + B_k \sin 2\phi$$
$$\bar{\theta} = A_\theta \cos 2\phi + B_\theta \sin 2\phi$$

the five equations (F.22.1), (I.2.1), (I.3.1), (F.11.2), and (F.23.2) for the nonsymmetric terms yield ten equations with eight unknowns A_q, B_q, A_s, B_s, A_w, B_w, A_θ, and B_θ; since b and k_2 are parameters (Pachner, 1971) A_b, B_b, A_k, B_k are also parameters. One pair of these ten equations is identically satisfied; the combination is

$$(\text{I.3.1}) + (\text{I.4.1}) - \frac{3}{2}\frac{d}{dt}(\text{I.4.1}) = (\text{I.2.1})$$

One pair of the equations determines A_θ and B_θ and the remaining three pairs of equations are

$$\begin{aligned} A_q &= A_s - \tfrac{2}{3}X \\ B_q &= B_s + \tfrac{2}{3}Y \end{aligned} \tag{2.2}$$

$$\begin{aligned} \dot{A}_s - 2\kappa(B_s - 2B_b + \tfrac{1}{3}Y)\,e^{-2\beta} &= 0 \\ \dot{B}_s + 2\kappa(A_s - 2A_b - \tfrac{1}{3}X)\,e^{-2\beta} &= 0 \end{aligned} \tag{2.3}$$

$$\begin{aligned} \ddot{X} + 2\dot{\beta}\dot{X} - [X + 3(A_k - A_s)]\,e^{-\alpha-2\beta} - 2\kappa\dot{Y} &= 0 \\ \ddot{Y} + 2\dot{\beta}\dot{Y} - [Y - 3(B_k - B_s)]\,e^{-\alpha-2\beta} + 2\kappa\dot{X} &= 0 \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} X &= A_W\,e^{-2\beta} \\ Y &= B_W\,e^{-2\beta} \end{aligned}$$

The pairs of equations (2.2), (2.3), and (2.4) completely determine the time-evolution of the nonsymmetric terms of the lowest powers in r. Once a solution of these equations is known, the nonsymmetric terms of the next higher powers in r (such as A_Π, B_Π, A_Σ etc.) can be obtained. It is well-known, Nariai (1970) and Miketinac & Pachner (1972), that such a simple iterative procedure does not exist for the symmetric terms. Therefore, in order to discuss the equations (2.2), (2.3), and (2.4) it will be necessary to assume a particular form for, say, β (all other symmetric terms can then be computed), which is typical for the physical situation under consideration. This is not difficult in the case of a collapsing cylinder. A cylinder collapses when distances between its particles tend to diminish. Because of equation (2.1) this means that $\beta \to -\infty$ when $t \to t_0$, t_0 being the moment at which the cylinder has collapsed. It can be taken that $t_0 = 1$. Letting $\beta(t = 0) = 0$, any other value simply changes the scale, a typical form for β is

$$\beta = \ln(1 - t) \tag{2.5}$$

It will be argued that no other form for β will change the conclusion that the nonsymmetric contribution to the density becomes insignificant as $t \to 1$.

3. *Nonsymmetric Contribution to the Density*

From the expression for the density (Pachner, 1971) it follows that

$$\frac{\rho}{\rho_i} = e^{-\alpha-2\beta}[1 + r^2(C_s + C_a\cos 2\phi + C_b\sin 2\phi) + \ldots]$$

where ρ_i is the density at the origin at $t = 0$ and

$$\begin{aligned} C_s &= k_2 - \tfrac{1}{2}(h + q + s + \kappa^2 e^{-2\beta}) \\ C_a &= A_k - \tfrac{1}{2}(A_q + A_s) \\ C_b &= B_k - \tfrac{1}{2}(B_q + B_s) \end{aligned}$$

It is assumed that at the initial moment all nonsymmetric terms vanish, but the density contains a nonsymmetric contribution

$$C_a = A_k \neq 0, \qquad B_k = 0$$

If there is no rotation (i.e. $\kappa = 0$) and $\dot{Y}(t = 0) = 0$, equations (2.2), (2.3), and (2.4) imply $A_s = B_s = Y = 0$ and since, then, $C_a = A_k + \frac{1}{3}X$ the equation for X takes the form

$$\ddot{C}_a = -2\dot{\beta}\dot{C}_a + C_a e^{-\alpha-2\beta} \tag{3.1}$$

Since $-\dot{\beta}$ and $e^{-\alpha-2\beta} \to \infty$ as $t \to 1$, it follows from (3.1) that $C_a \to \infty$ or $-\infty$ as $t \to 1$ depending on the choice of $C_a(0)$ and $\dot{C}_a(0)$. In the special case when β is given by (2.5) C_a was computed numerically (Fig. 1) assuming

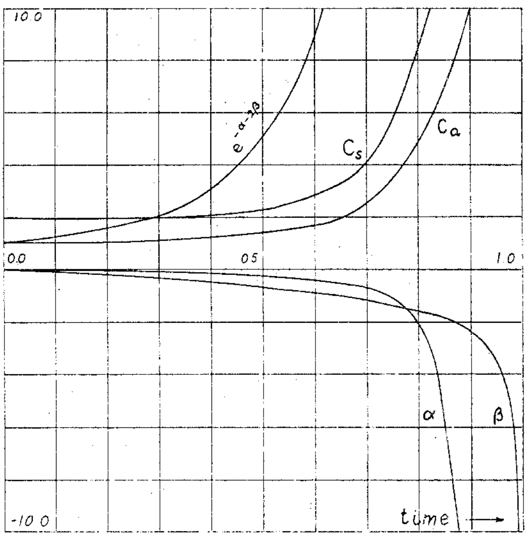

Figure 1.

$C_a(0) = 1$ and $\dot{C}_a(0) = 0$. To integrate (3.1) α must be computed from the equation

$$\ddot{\alpha} + \dot{\alpha}^2 = -e^{-\alpha-2\beta} - 2(\ddot{\beta} + \dot{\beta}^2) \tag{3.2}$$

(Pachner, 1971). In the figure α was computed taking $\alpha(0) = 0$ and $\dot{\alpha}(0) = 0$.

In order to calculate the symmetric contribution to the density, C_s, it is necessary to know h, q, and s. The equations for h and q can be found in Pachner (1971); they are

$$h = -e^{-\alpha} - (\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta})\, e^{2\beta} \tag{3.3}$$

$$q = 2h + 3s + 2e^{-\alpha} - \dot{\beta}(2\dot{\alpha} + \dot{\beta})\, e^{2\beta} \tag{3.4}$$

The equation for s (Miketinac & Pachner, 1972) can be written in the following form

$$\ddot{s} = -2\dot{\beta}\dot{s} + s\, e^{-\alpha-2\beta} + F \tag{3.5}$$

where

$$F = -\tfrac{3}{4}\ddot{h} - \tfrac{1}{2}(2\dot{\alpha} + \dot{\beta})\dot{h} + \tfrac{1}{2}(\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\alpha}\dot{\beta})h - \tfrac{3}{2}h^2 e^{-2\beta}$$
$$- \tfrac{1}{2}(1 + \tfrac{1}{2}h)e^{-\alpha-2\beta} + \tfrac{1}{2}e^{-2\alpha-2\beta} - \tfrac{1}{4}\dot{\beta}(2\dot{\alpha} + \dot{\beta})e^{-\alpha}$$

It is assumed that the parameter k_2 is equal to 1 and that at the initial moment $s = \dot{s} = 0$. In the special case when β is given by equation (2.5) the function F in equation (3.5) reduces to $-\tfrac{3}{4}[e^{-\alpha}/(1-t)^2]$. Comparing (3.1) and (3.5) will show that $-s \to \infty$ faster than $C_a \to \infty$ as $t \to 1$. Therefore, the nonsymmetric contribution, $C_a \cos 2\phi$, to the density becomes increasingly insignificant compared to the contribution due to the symmetric terms,

$$C_s = -2s + \tfrac{3}{2} + \tfrac{1}{2}e^{-\alpha} + \tfrac{1}{2}(1-t)\dot{\alpha}$$

as $t \to 1$. Of course, this conclusion would not be correct, if $\dot{C}_a(0)$ were assumed to be very much bigger than 1 (10^3 or bigger). Such an assumption does not seem physically interesting.

If β is not given by (2.5), the expression $\ddot{\beta} + \dot{\beta}^2$ in (3.2) will not be equal to zero but will be negligible compared to $e^{-2\beta}$ as $t \to 1$. This follows from the fact that the ratio $(\ddot{\beta} + \dot{\beta}^2)/e^{-2\beta}$ equals $e^{\beta}[(d^2/dt^2)e^{\beta}]$ where the positive function e^{β} is equal to 1 at $t = 0$ and tends to zero when $t \to 1$. Using this the function F in (3.5) simplifies to

$$-\tfrac{1}{4}\dot{\beta}^2 e^{-\alpha} - \tfrac{1}{2}e^{-\alpha-2\beta}$$

which, again, implies that $-s \to \infty$ faster than $C_a \to \infty$ as $t \to 1$ and this means that, in general, the symmetric contribution to the density, C_s, dominates. This conclusion is true, also, in the case when $\kappa \neq 0$ but is sufficiently small so that $\alpha \to -\infty$, when $\beta \to -\infty$. It may be possible to extend this result to the case of a finite body, but it will be necessary to introduce two more gravitational potentials, g_{12} and g_{13}, into the equations and all functions will have to be expanded in powers of z and r.

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